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Exploring causal reasoning for explainable resilience quantification

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While the concept of resilience inspired overall risk quantification, analysis and management, causal modeling and inference based mainly on directed labeled acyclic graphs (DAGs) changed the application of advanced statistical methods to a wide range of human and societal, engineering and natural sciences. The question is formulated how to relate both advanced approaches to draw mutual advantage of each other. This is conducted by extending direct causal chain-like linking of threat events to damage events by considering all possible causal relations. The focus is to motivate this generalization by resorting to the application of the total law of probability, the chain rule, commutation of events conditioned upon, as well as by introducing a binary operator to consider or not the additional dependencies introduced. Thus, for resilience quantification probabilities using random elements representing seed, intermediate and damage events are provided as joint probability function expansions using conditional probability factors that can be visualized with DAGs. This allows to apply weighted graph models to visualize as well as to compute overall risks, including under simplifying assumptions. Examples illustrate expressions with up to two intermediate event transition layers The approach is discussed and how it enables future interfacing between resilience engineering quantification and causal approaches, as well as the option of a recursive generation of a causality inspired resilience quantification expression.

Keywords: Causal modeling and inference, Resilience expansion and quantification, Chain rule and inclusion-exclusion principle of probability, Probabilistic graphical model, Risk based on probabilities of random variable measuring consequences, labeled directed acyclic graph.

1. Introduction

As resilience quantities in a narrow sense depend on systemic behavior in case of disruptions are ongoing or occurred as well as on the activation of one or several counter and recovery measures, related probabilistic expressions need to consider several events and conditional probabilities to take account of multiple possible event sequences.

The question arises how to represent and generate such resilience expressions to be able to quantify them. The idea of the article is to use causal graph models (Pearl 2022) in the form of labeled directed acyclic graphs (DAGs) (Pearl 1997) to represent each possible dependency between seed threat events, via intermediate events to final damage events. In particular, the ambition is to go beyond linear causal chain models as described in (Konila Sriram et al. 2019) (Häring et al. 2025) (Chakraborty, Caulfield, and Pym 2025) or causal ad-hoc models developed for application domains within causal toolchains (Qalaji et al. 2024).

2. Overview Approach

The methodology of the paper is structured as follows. In section 3 the problem statement is formalized by introducing random elements and their discrete finite values representing all event types mentioned in the introduction. It is shown in section 4 that the joint probability function of damage events and seed events can be expanded under certain assumptions to cover intermediate event sequences, if intermediate event sets form partitions of the event space.

Section 5 derives how simple directed event chains can be extended to cover more conditional events up to full chain rule expansions. Permutation of events and non-consideration or consideration of events regarding dependencies finally generates all possible DAGs with predefined seed threat and resilience-aware final damage event. Also, combinatorial numbers of expected numbers of DAGs and related intermediate events are provided.

Section 6 lists calculation examples that confirm the consistency of the formal expressions

and combinatorial expressions. It provides all DAGs considering no intermediate layers, one and two layers for sample application cases. It observes that identical DAGs, i.e. causal structural dependency graph models representing resilience-aware event chains, are generated repeatedly. It resolves double counting by combining several DAG-based models using weighted parameterized models.

Section 7 concludes hinting at the potential option to provide a recursive generation process of all DAG options.

3. Formalizing Application of Total Law of Probability Connecting Seed and Final Damage Events with Intermediate Events

Let X_i be N_X random elements and $\{x_{i\iota}\}_{\iota=0,1,\dots,N_{X,i}-1}$ their codomain sets, e.g., assumed to be discrete finite sets. Then we can define for $i = 0,1,\dots,N_X - 1$ the probabilities that an event $X_i = x_{i\iota}$ occurs as $P(X_i = x_{i\iota})$. Note that the value range attributed to a single random element can contain combinations of events for the same event type, for instance double cyberattack or attack using pre-demage. Similarly, this can also hold for the intermediate and final events.

Let for instance $X_{0=\text{threat}}$ be the random element that labels single distinguishable threat events, then $\{x_{0=\text{threat},\iota}\}_{\iota=0,1,\dots,N_{X,0}-1}$ is the set of threats. Let $X_{N_X-1=\text{damage}}$ be the random element that labels single distinguishable short- and long-term damage effect levels to cover risk and resilience perspectives, e.g., financial or environmental loss categories, then $\{x_{N_X-1=\text{damage},\iota}\}_{\iota=0,1,\dots,N_{N_X-1=\text{damage}}-1}$ is the set of possible damage levels to be considered to assess threats in terms of risk and resilience.

The probability that a certain initial threat and a given damage effect occur jointly reads

$$P\big(X_{1=\text{damage}} = x_{\text{damage},\iota_1} \cap X_{0=\text{threat}} = x_{\text{threat},\iota_0}\big) = P\big(X_{\text{damage}} = x_{\text{damage},\iota_1} | X_{\text{threat}} = x_{\text{threat},\iota_0}\big) \cdot P(X_{\text{threat}} = x_{\text{threat},\iota_0}) \quad (1)$$

for $\iota_0 = 0,1,\dots,N_{X,0} - 1$ and $\iota_1 = 0,1,\dots,N_{X,1} - 1$, using in the second line of (1) the conditional probability that the damage occurs given the threat event and the probability that the threat occurs. Each event in (1) is represented by a random element that takes one of its values.

In case a random element X_1 can be introduced such that

$$P\big(X_{2=\text{damage}} = x_{2,\iota_2} | X_{0=threat} = x_{threat,\iota_0}\big) = \sum_{\iota_1=0}^{N_{X,1}-1} P\,(X_{2=\text{damage}} = x_{2,\iota_2} | X_1 = x_{1,\iota_1}) \cdot P(X_1 = x_{1,\iota_1} | X_{0=\text{threat}} = x_{0,\iota_0}), \quad (2)$$

the following interpretation is feasible. The single threat events lead to single intermediate events $X_1 = x_{1,\iota_1}$, $\iota_1 = 0,1,\dots,N_{X,1} - 1$, that cause the final damage effects. The equality sign in (2) implies that this mapping between the random element codomain sets can be interpreted as an application of the total law of probability. For instance, in case of a cyberattack, the intermediate layer could be the software subsystems affected realizing the overall software functions.

Examples of possible expansions like (2) include random elements covering hazard source detailed descriptions, threat propagation descriptions, local loading, or physical-technical disruption including of network nodes that lead to final damage.

Inserting (2) into (1) allows already to compute whether a certain threat $x_{\text{threat},\iota_0}$, $\iota_0 = 0,1,\dots,N_{X,0} - 1$ causes a certain damage event $x_{\text{damage},\iota_2}, \iota_2 = 0,1,\dots,N_{X,N_X-1}$ considering one intermediate assessment layer, i.e. $N_X = 3$,

$$P\big(X_{2=\text{damage}} = x_{\text{damage},\iota_2} \cap X_{0=\text{threat}} = x_{\text{threat},\iota_0}\big) = \sum_{\iota_1=0}^{N_{X,1}-1} P\,\big(X_{2=\text{damage}} = x_{2,\iota_2} \big| X_1 = x_{1,\iota_1}\big) \cdot P\big(X_1 = x_{1,\iota_1} \big| X_{0=\text{threat}} = x_{0,\iota_0}\big) \cdot P(X_0 = x_{0,\iota_0}). \quad (3)$$

Equation (3) can be generalized by considering $N_X - 2$ intermediate layers to

determine damage effects from single threat events. In (3) one has $N_X = 3$ and there is $N_X - 2 = 3 - 2 = 1$ intermediate event layer. For later formalization convenience when identifying a special case of the product rule of probability, first equation (3) is rearranged as

$$P\left(X_{0=\text{threat}} = x_{\text{threat},\iota_0} \cap X_{2=\text{damage}} = x_{\text{damage},\iota_2}\right) = P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \sum_{\iota_1=0}^{N_{X,1}-1} P\left(X_1 = x_{1,\iota_1} \middle| X_{0=\text{threat}} = x_{0,\iota_0}\right) \cdot P\left(X_{2=\text{damage}} = x_{2,\iota_2} \middle| X_1 = x_{1,\iota_1}\right). \quad (4)$$

Assuming $N_X - 2$ intermediate propagation layers with X_1 to X_{N_X-2}, equation (4) generalizes to

$$P\left(\begin{array}{c} X_{0=\text{threat}} = x_{0,\iota_0} \\ \cap\ X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \end{array}\right) = P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \sum_{\iota_1=0}^{N_{X,1}-1} P\left(X_1 = x_{1,\iota_1} \middle| X_{0=\text{threat}} = x_{0,\iota_0}\right) \cdot \sum_{\iota_2=0}^{N_{X,2}-1} P\left(X_2 = x_{2,\iota_2} \middle| X_1 = x_{1,\iota_1}\right) \quad \cdots \cdot \sum_{\iota_{N_X-3}=0}^{N_{X,N_X-3}-1} P\left(X_{N_X-3} = x_{N_X-3,\iota_{N_X-3}} \middle| X_{N_X-4} = x_{N_X-4,\iota_{N_X-4}}\right) \cdot \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} P\left(X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}} \middle| X_{N_X-3} = x_{N_X-3,\iota_{N_X-3}}\right) \cdot P\left(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \middle| X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}}\right). \quad (5)$$

To see that (5) is a sum over simple causal chains the sum signs are collected resulting in

$$P\left(\begin{array}{c} X_{0=\text{threat}} = x_{0,\iota_0} \\ \cap\ X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \end{array}\right) = \sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} P(X_{0=\text{threat}} = x_{0,\iota_0}) \cdot P\left(X_1 = x_{1,\iota_1} \middle| X_{0=\text{threat}} = x_{0,\iota_0}\right) \cdot P\left(X_2 = x_{2,\iota_2} \middle| X_1 = x_{1,\iota_1}\right) \cdots \cdot P\left(X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}} \middle| X_{N_X-3} = x_{N_X-3,\iota_{N_X-3}}\right) \cdot P\left(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \middle| X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}}\right) = \sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \prod_{k=1}^{N_X-2} P\left(X_k = x_{k,\iota_k} \middle| X_{k-1} = x_{k-1,\iota_{k-1}}\right) \cdot P\left(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \middle| X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}}\right), \quad (6)$$

where the last equation in (6) uses the product operator to cover in a compact way the intermediate factors of all conditional probabilities. All the factors for each combination of the integer index sets $((1,\iota_1),(2,\iota_2),\ldots,(N_X-2,\iota_{N_X-2}))$ form a causal directed chain.

The number of causal chains in (6) reads according to the sum ranges

$$N_C = \prod_{k=1}^{N_X-2} N_{X,k}. \quad (7)$$

4. Complete Knowledge of Resilience Using Permutations of the Chain Rule Expression and Possible Binary Simplifications

A generalization of the causal chain dependency of the intermediate terms as in (6) is first to assume that instead of the causal chain dependency the chain rule holds. This means each factor keeps all dependencies on the prior events instead of only keeping the dependency on the last event as in (6) resulting in

$$\sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \prod_{k=1}^{N_X-2} P\left(X_k = x_{k,\iota_k} | X_{k-1} = x_{k-1,\iota_{k-1}} \cap X_{k-2} = x_{k-2,\iota_{k-1}} \cap \cdots \cap X_1 = x_{1,\iota_1} \cap X_{\text{threat}=0} = x_{0,\iota_0}\right) \cdot P\left(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \middle| X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}} \cap X_{N_X-3} = x_{N_X-3,\iota_{N_X-3}} \cap \cdots\right), \quad (8)$$

$$\cap X_1 = x_{1,\iota_1} \cap X_{\text{threat}=0} = x_{0,\iota_0}\big).$$

$$= \sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1}$$

$$\mathrm{P}\big(X_{\text{threat}=0} = x_{0,\iota_0} \cap X_1 = x_{1,\iota_k} \cap \cdots$$

$$\cap X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}}$$

$$\cap X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}}\big),$$

where the last equality sign in (8) used the definition of the chain rule for the probability of the N_C combinations of the intermediate events

$$X_1 = x_{1,\iota_k} \cap \cdots \cap X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}} \quad (9)$$

$$\cap X_{N_X-2} = x_{N_X-2,\iota_{N_X-2}}.$$

Even a further generalization of (6) beyond (8) is feasible if all permutations of the events in the right-hand side of (8) are allowed but the initial and final event representing all possible orderings of the intersections of the events in (9). This is achieved by considering all possible $(N_X-2)!$ permutations of the N_X-2 intermediate events in each addend of (8). Equation (9) is generalized to generate all possible intermediate probability chain rule expressions resulting in

$$\frac{1}{(N_X-2)!} \sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} \sum_{\sigma\in\Sigma^{N_X-2}}$$

$$\mathrm{P}\big(X_{\text{threat}=0} = x_{0,\iota_0} \cap X_{\sigma(1)} = x_{\sigma(1,\iota_k)} \cap \cdots$$

$$\cap X_{\sigma(N_X-2)} = x_{\sigma(N_X-2,\iota_{N_X-2})} \quad (10)$$

$$\cap X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}}\big)$$

$$= \frac{1}{(N_X-2)!} \sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} \sum_{\sigma\in\Sigma^{N_X-2}}$$

$$P(X_{\text{threat}=0} = x_{0,\iota_0})$$

$$\cdot \prod_{k=1}^{N_X-2} P\big(X_{\sigma(k)} = x_{\sigma(k,\iota_k)} | X_{\sigma(k-1)} = x_{\sigma(k-1,\iota_{k-1})}$$

$$\cap X_{\sigma(k-2)} = x_{\sigma(k-2,\iota_{k-1})} \cap \cdots \cap X_{\sigma(1)} = x_{\sigma(1,\iota_1)}$$

$$\cap X_{\text{threat}=0} = x_{0,\iota_0}\big) \quad (11)$$

$$\cdot P\left(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \middle|\right.$$

$$\left. X_{\sigma(N_X-2)} = x_{\sigma(N_X-2,\iota_{N_X-2})}\right.$$

$$\cap X_{\sigma(N_X-3)} = x_{\sigma(N_X-3,\iota_{N_X-3})} \cap \cdots$$

$$\cap X_{\sigma(1)} = x_{\sigma(1,\iota_1)} \cap X_{\text{threat}=0} = x_{0,\iota_0}\big),$$

where Σ^{N_X-2} is the set of all possible permutations of the N_X-2 different indices $1,2,\ldots,N_X-2$, which act on the composed indices $(1,\iota_1),(2,\iota_2),\ldots,(N_X-2,\iota_{N_X-2})$ as already introduced after (4). The number of permutations is $N_\sigma = |\Sigma^{N_X-2}| = |\{\sigma^{N_X-2}\}| = |\{\sigma\}| = (N_X-2)!$. The left-hand side of (11) shows that the permutations act on all indices but of the initial and final events, which is taken up in the conditional probabilities in the right-hand side of (11).

The normalization factor at the left-hand side of (11) takes account of the equivalency of all possible chain rule expansions of the right and side of (8).

Using (7) and (9) there are

$$N_{CC} = (N_X-2)! \prod_{k=1}^{N_X-2} N_{X,k} \quad (12)$$

different causal chain options that connect each of the $N_{X,0=\text{threat}}$ possible threat events with each of the possible $N_{X,N_{N_X-1=\text{damage}}}$ final damage events. Considering all combinations leads to

$$N_{ACC} = N_{X,0=\text{threat}}\, N_{X,N_{N_X-1=\text{damage}}} \cdot (N_X-2)! \prod_{k=1}^{N_X-2} N_{X,k}\,. \quad (13)$$

It is now observed that in (6) a simplified version of (11) is considered where only one dependency is assumed on which the events are conditioned resulting in the causal chain like structure. This arises the question, if a general mechanism of allowing conditioning on events can be applied to the right-hand side of (11).

As preparation, the number of events on which on the right-hand side of (11) is conditioned upon is determined. The 1-st factor has zero events on which it is conditioned upon. The 2-nd factor corresponding to $k=1$ in the right-hand side of (11), $P\big(X_1 = x_{\sigma(1,\iota_1)} | X_0 = x_{0,\iota_0}\big)$ is conditioned on 1 event, which is not permutated. The 3-rd factor $P\big(X_2 = x_{\sigma(2,\iota_2)} | X_1 = x_{\sigma(1,\iota_1)} \cap X_0 = x_{0,\iota_0}\big)$ is conditioned on 2 events out of which 1 is permutated and the last event is not permutated. Hence, the conjecture is that the n-th factor for $n \geq 2$ is conditioned on $n-1$ events out of which $n-2$ are permutated. To check this, we inspect the second last factor of the right-hand side of (11) and find that the (N_X-1)-th factor

corresponding to $k = N_X - 2$ in (11) is conditioned on $N_X - 2$ events out of which $N_X - 3$ are permutated. Finally, the last and N_X-th factor is conditioned on $N_X - 1$ events out of which $N_X - 2$ are permutated.

Now adding up the total number of events on which it is conditioned upon in (11) results in

$$N_C = 1 + 2 + \cdots + N_X - 1 = \frac{N_X(N_X-1)}{2} = \binom{N_X}{2} = \frac{N_X!}{(N_X-2)!\,2!}, N_X \geq 2. \quad (14)$$

Note that for $N_X = 2$ no intermediate random element is inserted, nevertheless it is conditioned on the single selected random threat element value, see (14).

In total each conditional dependency option as counted in (14) based on (11) can be lifted or not. Thus, the total number of binary operations of removing or not removing a conditional dependency in the last expression of (11) reads

$$N_B = 2^{N_C} = 2^{\frac{N_X(N_X-1)}{2}}. \quad (15)$$

Extending (13), the total number of causal chains reads (removing or not removing events on which one conditions upon as well as allowing for a permutation of all intermediate events)

$$N_{TCC} = N_{X,0=\text{threat}}\ N_{X,N_{N_X-1=\text{damage}}}\ 2^{\frac{N_X(N_X-1)}{2}} \cdot (N_X-2)! \prod_{k=1}^{N_X-2} N_{X,k}\,. \quad (16)$$

The interpretation provides a direct insight in the increase of complexity of resilience computations as all factors after the first two of (16) can be interpreted as being due to the consideration of additional intermediate layers between threat and final effect on citizens, economy, environment, etc. While the last product operator factor in (16) just counts the options of intermediate event chains assuming only simple causal chain dependencies, the second last factorial factor considers permutations of the latter and the third considers all simplifications of the expanded chain-rule-like dependencies. The exponential factor dominates the computation of N_{TCC}.

Whereas (16) counts the maximum number of expected connections between threat and damage event, an explicit construction rule of all possible dependency structures reads

$$\sum_{\iota_1=0}^{N_{X,1}-1} \sum_{\iota_2=0}^{N_{X,2}-1} \cdots \sum_{\iota_{N_X-2}=0}^{N_{X,N_X-2}-1} \sum_{\sigma\in\Sigma^{N_X-2}} \sum_{B\in\mathrm{B}^{N_C}}$$

$$P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \prod_{k=1}^{N_X-2} P\big(X_{\sigma(k)} = x_{\sigma(k,\iota_k)} \big| B(X_{\sigma(k-1)} = x_{\sigma(k-1,\iota_{k-1})}) \cap B(X_{\sigma(k-2)} = x_{\sigma(k-2,\iota_{k-1})}) \cap \cdots \cap B(X_{\sigma(1)} = x_{\sigma(1,\iota_1)}) \cap B(X_{\text{threat}=0} = x_{0,\iota_0})\big) \quad (17)$$

$$\cdot P\Big(X_{N_X-1=\text{damage}} = x_{N_X-1,\iota_{N_X-1}} \Big| B(X_{\sigma(N_X-2)} = x_{\sigma\left(N_X-2,\iota_{N_X-2}\right)}) \cap B(X_{\sigma(N_X-3)} = x_{\sigma\left(N_X-3,\iota_{N_X-3}\right)}) \cap \cdots \cap B\big(X_{\sigma(1)} = x_{\sigma(1,\iota_1)}\big) \cap B(X_{0=threat} = x_{0,\iota_0})\Big),$$

where in (17) B is out of the set of binary operators acting on N_B elements labeled as B^{N_B}.

The action of B on a single event can be formalized as follows

$$B(X_i = x_{\sigma(i,\iota_i)}) = \begin{cases} X_i = x_{\sigma(i,\iota_i)}, \\ \text{event with label } i \text{ is considered} \\ \text{(no operator)}, \\ \text{event with label } i \text{ is not considered} \end{cases}. \quad (18)$$

The binary operator B as defined in (18) acts on all event expressions as written in (17) on which probabilities are conditioned upon.

A single operator B can be represented for the in total N_B elements it is acting upon using zeros and ones, e.g.,

$$B = (0,1,1,\cdots,0,1), \quad (19)$$

where zero means event with the corresponding position in (17) is not considered and one it is considered. For instance, all binary operators starting with zero at first position do *not* consider in the first conditional factor in (17), $P\big(X_{\sigma(1)} = x_{\sigma(1,\iota_1)} | B(X_0 = x_{0,\iota_0})\big)$, the dependency.

5. Application of causal resilience dependency expansions and combination options

5.1. *Example with no intermediate event layer*

For simplicity we consider the following setting in (17) $N_X = 2$, $N_{X,0=\text{threat}} = 2$, i.e., e.g.,

$X_0 = x_{o,\iota_0} \in \{x_{0,0} = 0 = \text{"cyberattack"},$ $\mathrm{x}_{0,1}$=1="flooding"$\}$ and $N_{X,1=\text{damage}} = 3$,

e.g. $X_1 = x_{1,\iota_1} \in \{x_{1,0} = 0 = \text{"injuries"}, x_{1,1}=1=\text{"few fatalities"}, x_{1,2}=2=\text{"many fatal."}\}$. Then the number of intermediate events that can be permuted reads $N_\sigma = (N_X - 2)! = 0! = 1$. The number of events on which it is conditioned upon in (17) according to (14) is $N_C = \binom{N_X}{2} = \binom{2}{2} = 1$. Hence, there are according to (15) only $N_B = 2^{N_C} = 2^1 = 2$ options for the binary operator acting on a single event, i.e., event considered or not.

Using this setting, (17) generates the following conditional probability product chains

$$\sum_{\sigma\in\Sigma^{N_X-2=0}=id.} \sum_{B\in B^{N_C=2}=\{(0),(1)\}} P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot P\left(X_{N_X-1=2-1=1=\text{damage}} = x_{N_X-1=1,\iota_{N_X-1=1}} \middle| B(X_{N_X-2=2-2=0=\text{threat}} = x_{N_X-2=0,\iota_{N_X-2=0}})\right) \tag{20}$$

$$= \sum_{B\in B^{N_C=2}=\{(0),(1)\}} P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot P\left(X_{1=\text{damage}} = x_{1,\iota_1} \middle| B(X_{0=\text{threat}} = x_{N_X-2=0,\iota_{N_X-2=0}})\right)$$

$$= P(X_{\text{threat}=0} = x_{0,\iota_0})\, P(X_{1=\text{damage}} = x_{1,\iota_1}) + P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot P\left(X_{1=\text{damage}} = x_{1,\iota_1} \middle| X_{0=\text{threat}} = x_{0,\iota_0}\right),$$

which holds for $(0,\iota_0) \in \{(0,\iota_0 = 0), (1,\iota_0 = 1 = N_{X,0} - 1)\}$, and $(1,\iota_1) \in \{(N_{X,1=N_X-1=1} = 1, \iota_1 = 0), (1,\iota_2 = 1), (1,\iota_2 = N_{X,1} - 1 = 2)\}$. Hence, there are in total $2\, N_{X,0}\, N_{X,1}$ options, which agrees well with (16) according to which we have the same value $N_{TCC} = N_{X,0}\, N_{X,1}\, 2^{\frac{2(2-1)}{2}} 0! \prod_{k=1}^{0} N_{X,k} = N_{X,0}\, N_{X,1}\, 2$ as the last two factors equal 2.

In (20) we use that there are no intermediate events, which formally arises as $N_X - 2 = 2 - 2 = 0 < 1$. The product factor of (17) is also not considered as there are no intermediate events, formally as $k = 1 > N_X - 2 = 2 - 2 = 0$, resulting in an empty product. Similarly, the last conditional probability expression in the right-hand side of (17) reduces to a dependency on a single event only.

For the first equality sign in (20) we use that there is only one permutation for one number, namely the identity $id.$ to omit the summation over permutations.

Note that the first expression of the right hand side of (20) is just the classical definition of risk, whereas the second term even generates the classical definition of vulnerability in terms of conditioning the consequences on the occurrence of the damage event.

5.2. *Example with one intermediate event layer*

Now we set in (17) $N_X = 3$, $N_{X,0=\text{threat}} = 2$ with values as in section 1, $N_{X,1} = 2$, e.g. $X_1 = x_{1,\iota_1} \in \{x_{1,0} = 0 = \text{"system damage"}, x_{1,1}=1=\text{"major system damage"}\}$ and $N_{X,2=\text{damage}} = 3$, e.g. with X_2 as X_1 in section 5.1. Then the number of intermediate random elements are evaluated is $N_X - 2$=1. The number of permutations considered reads $N_\sigma = (N_X - 2)! = 1$. The number of events on which it is conditioned upon in (17) according to (14) is $N_C = \binom{3}{2} = 3$. Hence, there are according to (15) already $N_B = 2^{N_C} = 2^3 = 8$ options for the binary operator.

Using these numbers in (17) we find subsequently

$$\sum_{\iota_1=0}^{N_{X,1}-1=1} \sum_{\sigma\in\Sigma^{N_X-2=1}=\{id.\}} \sum_{B\in B^{N_C=3}} P(X_{\text{threat}=0} = x_{0,\iota_0}) \cdot \prod_{k=1}^{N_X-2=1} P\left(X_{\sigma(k)} = x_{\sigma(k,\iota_k)} \middle| B(X_{\text{threat}=0} = x_{0,\iota_0})\right) \cdot P\left(X_{N_X-1=2=\text{damage}} = x_{N_X-1=2,\iota_2} \middle| B(X_{\sigma(N_X-2=1)} = x_{\sigma(N_X-2=1,\iota_1)}) \cap B(X_{0=\text{threat}} = x_{0,\iota_0})\right) \tag{21}$$

$$= \sum_{\iota_1=0}^{N_{X,1}-1=1} \sum_{B\in \mathrm{B}^{N_C=3}=\{(0,0,0),(0,01),(0,1,0),\cdots,(1,1,1)\}}$$

$$P(X_{\text{threat=0}} = x_{0,\iota_0})$$

$$\cdot P\big(X_1 = x_{1,\iota_1} | B(X_{\text{threat=0}} = x_{0,\iota_0})\big)$$

$$\cdot P\big(X_{2=\text{damage}} = x_{2,\iota_2} \big| B(X_1 = x_{1,\iota_1}) \cap B(X_{0=\text{threat}} = x_{0,\iota_0})\big)$$

$$= P(X_{\text{threat=0}} = x_{0,\iota_0})\, P(X_1 = x_{1,0}) \cdot P\big(X_{2=\text{damage}} = x_{2,\iota_2}\big)$$

$$+P(X_{0=\text{threat}} = x_{0,\iota_0})\, P(X_1 = x_{1,0}) \cdot P\big(X_{2=\text{damage}} = x_{2,\iota_2} \big| X_{0=\text{threat}} = x_{0,\iota_0}\big)$$

+ [8-2-1=5 more terms]

$$+P(X_{0=\text{threat}} = x_{0,\iota_0})\, P\big(X_1 = x_{1,0} | X_0 = x_{0,\iota_0}\big) \cdot P\big(X_{2=\text{damage}} = x_{2,\iota_2} \big| X_1 = x_{1,0} \cap X_{0=\text{threat}} = x_{0,\iota_0}\big)$$

+ [8 terms as above with $x_{1,0}$ replaced by $x_{1,1}$].

The first equality sign in (21) uses that permutations are only allowed for a single intermediate event index. In total 16 different event chains were generated in (21). According to (16) there are $N_{TCC} = N_{X,0=threat}\ N_{X,N_{2=damage}}\ 2^{\frac{3(3-1)}{2}}(3-2)!\prod_{k=1}^{3-2} N_{X,k} = 2\cdot 3\cdot 2^3\cdot 1\cdot 2 = 6\cdot 16$ different combinations for all threat seed events and potential damage effects. This confirms that (21) computed all potential causal dependencies for a single combination of threat and damage event.

5.3. *Example with two intermediate event layers*

Intermediate additional layer random element's values cover, e.g., the number of failed nodes of a network, for instance in case of only 2 nodes one has the set $\{\emptyset, 0{,}1, (0{,}1)\}$, which is attributed to X_1 with $N_{X,1} = 4$. All other random elements are as in section 5.2. Using (17) the following events are generated:

$$\sum_{\iota_1=0}^{N_{X,1}-1=3} \sum_{\iota_{N_X-2=4-2=2}=0}^{N_{X,N_X-2=4-2=2}-1=1} \sum_{\sigma\in\Sigma^{N_X-2=4-2=2}=\{\text{id.,exchange}\}} \sum_{B\in \mathrm{B}^{N_C=\binom{4}{2}=6}=\{(0,0,0,0,00),(0,0,0,0,0,1),\cdots,(1,1,1,1,1,1)\}}$$

$$P(X_{\text{threat=0}} = x_{0,\iota_0}) \qquad (22)$$

$$\cdot \prod_{k=1}^{N_X-2=4-2=2} P\big(X_{\sigma(k)} = x_{\sigma(k,\iota_k)} | B(X_{\sigma(k-1)} = x_{\sigma(k-1,\iota_{k-1})}) \cap B(X_{\text{threat=0}} = x_{0,\iota_0})\big)$$

$$\cdot P\Big(X_{N_X-1=4-1=3=\text{damage}} = x_{N_X-1=3,\iota_{N_X-1=3}} \Big| B(X_{\sigma(N_X-2=4-2=2)} = x_{\sigma\left(N_X-2=2,\iota_{N_X-2=2}\right)}) \cap B(X_{\sigma(N_X-3=1)} = x_{\sigma\left(N_X-3=1,\iota_{N_X-3=1}\right)}) \cap B(X_{0=\text{threat}} = x_{0,\iota_0})\Big)$$

$$= P(X_{\text{threat=0}} = x_{0,\iota_0})\, P(X_1 = x_{1,0}) \cdot P\big(X_2 = x_{2,0}\big) P(X_{3=\text{damage}} = x_{3,\iota_3})$$

$$+ P(X_{\text{threat=0}} = x_{0,\iota_0})\, P(X_1 = x_{1,0}) \cdot P\big(X_2 = x_{2,0}\big)\, P(X_{3=\text{damage}} = x_{3,\iota_3} | X_0 = x_{0,\iota_0})$$

+ [2^6-2=62 more terms with increasingly more conditional dependencies]

$$+ P(X_{\text{threat=0}} = x_{0,\iota_0})\, P(X_2 = x_{2,0}) \cdot P\big(X_1 = x_{1,0}\big) P(X_{3=\text{damage}} = x_{3,\iota_3})$$

$$+ P(X_{\text{threat=0}} = x_{0,\iota_0})\, P(X_2 = x_{2,0}) \cdot P\big(X_1 = x_{1,0}\big)\, P(X_{3=\text{damage}} = x_{3,\iota_3} | X_0 = x_{0,\iota_0})$$

+ [2^6-2=62 more terms with increasingly more conditional dependencies]

+ [$2\cdot 64\ (N_{X,1}\, N_{X,2} - 1)$ more terms as just listed considering all remaining options of combinations of the first two sum operators].

The number of binary operators is correctly counted in (22) as in the product expression there are 3 operators leading in total to $N_B = 6$ binary operators in accordance with (14). Note that the first factor in the last product expression in (17) is conditioned only on one event.

In total in (21) there are $2\cdot 64\, N_{X,1}\, N_{X,2}$ expressions that are added up, which is in accordance with (16) when counting only the options for one combination of threat and damage effect as $2^{\frac{N_X(N_X-1)}{2}}(N_X-2)!\prod_{k=1}^{N_X-2} N_{X,k} = 2^{4\cdot 3/2}\cdot 2!\ N_{X,1}\, N_{X,2} = 64\cdot 2\, N_{X,1}\, N_{X,2}$.

5.4. *Avoiding double counting of event chains*

Note that, e.g., the 1-st and the 65-the addend in the last sum of (22) are identical, as the

probabilities in the products can be permuted. Hence, the (17) leads to some additional expressions, which need to be subtracted from the sum if it is defined to contain only different causal connections of seed events and damage events.

Avoidance of double counting can formally be enforced requiring that each additional term is different from each already identified term. Or by forming a set out of the addends of (17).

As each addend in (17) corresponds to a single potential causal model relating the seed and the final events, which need to be assessed and parametrized based on data, identification of the same event chains is easily feasible in practice.

5.5. *Weighted sum of dependency model options*

In practice, a feasible approach is to weight possible causal relations regarding their relative expected importance for a connection of seed and damage events while considering the weighted sum represents a joint normalized probability function as defined in (1). Such weights, which even can be set zero for identical labeled DAGs to ensure uniqueness, can be inserted in (17) before each addend, e.g., using the notation

$$w_{\iota_0,\iota_1,\cdots,\iota_{N_X-2},\iota_{N_X}-1}(\sigma, B). \quad (23)$$

The causal structural chains as defined in (17) can be visualized using graphical labeled directed acyclic graph (DAG) representations, where probabilities of events without conditioning are just nodes. Conditioning of an event is represented by an arrow where the arrow tip points to the event that is conditioned, and the starting point of the arrow originates from a node representing the event on which it is conditioned. Thus, a single DAG contains all information on conditioning for a single addend of (17).

6. Conclusions

The paper motivated and presented a generation process of causal structural graph like dependencies to connect seed threat events with resilience-aware damage events considering all possible intermediate events of intermediate layers represented by random elements and their values. Thus, causal structural hypotheses are generated in the form of causal graph models covering all possible causal relations.

This allows to represent resilience quantities as weighted sums of dependency relations between events represented by random elements that can be visualized as DAGs, thus generating a systematic dependency importance expansion.

The final generation algorithm does not assume that the intermediate events need to occur in a predefined order. For instance, intermediate events can be observations on grid nodes of a network as collected by different health monitoring systems, measurements at different times, or human assessments of overall system health, for which multiple dependency relations are conceivable.

Whereas the generation process is rather straightforward, it was found to generate also identical causal graph models. This could be overcome in future work by using an inclusion-exclusion principle like construction rule. However, it is expected to be less intuitive and only to allow a recursive formulation, similarly as used in the proof for the number of different possible DAGs for n nodes (OISE 2026).

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